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Eigenfrequencies of tapered rectangular plates with intermediate line supports

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Abstract

The free vibrations of a wide range of tapered rectangular plates with an arbitrary number of intermediate line supports in one or two directions are investigated. The domain of the plate is bounded by $x = \alpha a$, $a(0 \leqslant \alpha < 1)$ and $y = \beta b$, $b(0 \leqslant \beta < 1)$ in the rectangular co-ordinates. The thickness of the plate is continuously varying and proportional to a power function $x^s y^t$. A variety of tapered rectangular plates can be described by giving the taper factors s and t various values. The intermediate line supports run parallel to the edges of the plate. A new set of admissible functions, which are the static solutions of the tapered beam with intermediate point supports, or a strip taken from the plate structure in one or the other direction under a Taylor series of loads, is developed. Consistent convergency independent of the truncation factors α and β of the plate can be obtained by taking the midpoint of the beam as the expanding point of the Taylor series. Unlike conventional admissible functions, this set of static beam functions can appropriately vary with the thickness variation of the plate. The eigenfrequency equation of the plate is derived by the Rayleigh–Ritz approach. A general computer program has been compiled. It can be seen that the convergency of the numerical computation is very rapid and that the first few eigenfrequencies can be obtained with good accuracy by using only a small number of terms of the static beam functions. Sets of first-time reported eigenfrequency data are included for future reference.

1. Introduction

The rectangular plate is one of the most widely used structural elements in engineering. In practical applications, intermediate line supports may be placed to reduce the magnitude of dynamic and static stresses and displacements of the structure or satisfy special architectural and functional requirements. It is important for the designers to understand the effect of intermediate line supports on the dynamic behavior of the structure.

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Leissa (1973) summarized the research work on the vibration of rectangular plates before the early 1970's. Many of the investigations were about the uniform rectangular plates. The problem of the vibration of plates with varying complexity has received rather less attention. Appl and Byers (1965) analysed the fundamental frequency of a rectangular plate with all edges simply supported and with linear thickness variation in one direction by using the method of upper and lower bounds. Kobayashi and Sonoda (1991) applied the power series expansions to analyse the vibration and buckling of rectangular plates with two opposite edges simply supported and linearly varying thickness in one direction and Bert and Malik (1996) used the differential quadrature method to investigate such plates. Soni and Sankara Rao (1974) analysed the free vibration of rectangular plates having two opposite edges simply supported and exponentially varying thickness in one direction by using a quintic spline technique of solution. Pulmano and Gupta (1976) used the finite strip method to analyse the free vibration of linearly tapered rectangular plates. Bhat et al. (1990) used the one-dimensional orthogonal polynomials to study a one-direction, linearly tapered rectangular plate with different combinations of boundary conditions, aspect ratio and truncation factor, and the fundamental frequency coefficient was also determined, with excellent accuracy, by means of the optimized Kantorovich method proposed by Laura and Cortinez (1988). Dawe (1966) analysed the free vibration of rectangular plate with general variable thickness by the use of the finite element method.

If intermediate supports are added to the plate, the vibratory characteristics of the structure will change accordingly. Elishakoff and Sternberg (1979) used the modified Bolotin's method and Azimi et al. (1984) used the receptance method to analyse the free vibration of rectangular plates simply supported at two opposite edges and continuous over line supports perpendicular to those edges. Takahashi and Chishaki (1979) presented a sine series solution for the free vibration of simply supported rectangular plates over a number of line supports in two directions. Zhou (1994) used a set of modified vibrating beam functions, Kim and Dickinson (1987) used a set of one-dimensional orthogonal polynomials and Liew and Lam (1991) use a set of two-dimensional orthogonal polynomials to analyse the free vibration line supported rectangular plates in one and two directions by the Rayleigh–Ritz method, and Cheung and Kong (1995) applied the finite strip method to analyse such plates.

The vibration analysis of tapered rectangular plates with intermediate line supports is not yet available in the current literature. This may be due to the difficulty in forming a simple and adequate deflection function which can apply to the entire plate domain and satisfy both the boundary conditions and the intermediate support conditions. This study attempts to fill this apparent void by providing sets of first-time presented eigenfrequency data for such plates. In this paper, the thickness of the plate to be considered is continuously varying and proportional to a power function $x^s y^t$ which may describe a wide range of tapered rectangular plates properly by varying the values of the taper factors s and t. A new set of admissible functions are developed from the static solutions of a tapered beam with intermediate point supports under an arbitrary static load which is expanded into a Taylor series. The beam is a unit width of strip taken from the tapered rectangular plate in the longitudinal direction or the vertical direction. Only a set of the static beam functions in some direction should be derived because the tapered plate considered is with a similar thickness variation in two directions. The Rayleigh–Ritz method is utilized to obtain the eigenfrequency equation of the plate. It is demonstrated that consistent and rapid convergency can be achieved for arbitrary truncation factors of the plate and that the first few eigenfrequencies may be obtained with good accuracy by using only a small number of terms of the static beam functions.

2. The Rayleigh–Ritz method for tapered rectangular plates

A tapered rectangular plate with an arbitrary number of intermediate line supports, as shown in Fig. 1, lies in the x–y plane and is bounded by edges $x = \alpha a$, a and $y = \beta b$, b where α ($0 \leqslant \alpha < 1$) and β ($0 \leqslant \beta < 1$) are referred to as truncation factors of the plate in the x and y directions, respectively. The truncated plate is part of the sharp ended plate. The side lengths of the plate are A and B in the x and y directions, respectively, where $A = (1-\alpha)a$ and $B = (1-\beta)b$. If the plate is with a sharp edge in the x direction then $\alpha = 0$ and if the plate is with that in the y direction then $\beta = 0$. There are K_x and K_y intermediate line supports acting on the plate in the x and y directions, respectively. The co-ordinates of the line supports in the x and y directions are x_k ($k = 1, 2, \ldots, K_x$) and y_k ($k = 1, 2, \ldots, K_y$), respectively. It is clear that $\alpha a < x_k < a$ ($k = 1, 2, \ldots, K_x$) and $\beta b < y_k < b$ ($k = 1, 2, \ldots, K_y$). It is assumed that the thickness $h(x,y)$ of the plate is described by a power function

$$h(x,y) = h_0 (x/a)^s (y/b)^t \tag{1}$$

where h_0 is the thickness of the plate at the point $x = a$, $y = b$. s and t are referred to as taper factors of the plate in the x and y directions, respectively. A variety of tapered plates can be described by giving the taper factors s and t values and some common tapered rectangular plates are shown in Table 1. The flexural rigidity of the plate is

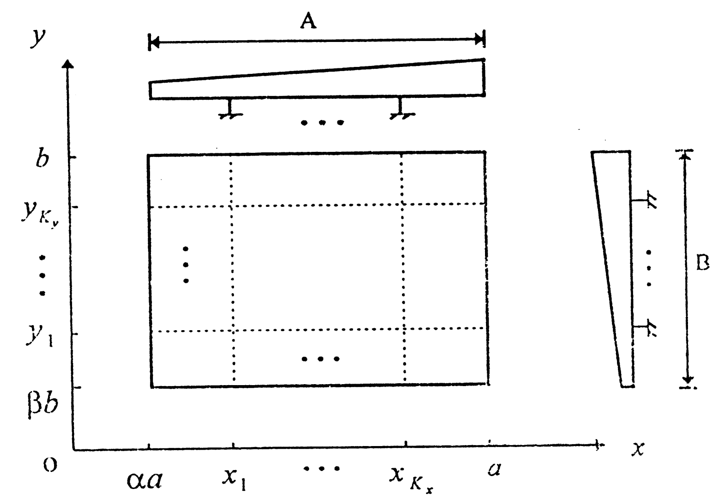

Fig. 1. A tapered rectangular plate with intermediate line supports.

Table 1
Some common rectangular plates with variable thickness

Type of non-uniform rectangular plates	Taper factors
A uniform plate	$s = 0$, $t = 0$
A linearly tapered plate in the x direction	$s = 1$, $t = 0$
A linearly tapered plate in the y direction	$s = 0$, $t = 1$
A linearly tapered plate in both directions	$s = 1$, $t = 1$
A parabolically tapered plate in the x direction	$s = 2$, $t = 0$
A parabolically tapered plate in the y direction	$s = 0$, $t = 2$
A parabolically tapered plate in both directions	$s = 2$, $t = 2$

$$D(x,y) = D_0(x/a)^{3s}(y/b)^{3t} \tag{2}$$

in which, $D_0 = Eh_0^3/12(1-\nu^2)$ where E is the Young's modulus and ν is the Poisson's ratio.

Assuming that the largest thickness of the plate is small compared to its boundary dimensions and that the classical plate theory is valid, the maximum strain energy $U_{\max}$ and the maximum kinetic energy $T_{\max}$ of the plate are given by

$$U_{\max} = \frac{1}{2}\int_{\alpha a}^{\alpha}\int_{\beta b}^{b} D(x,y)\left\{\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + 2\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + \left(\frac{\partial^2 W}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} - \left(\frac{\partial^2 W}{\partial x\,\partial y}\right)^2\right]\right\}\mathrm{d}y\,\mathrm{d}x, \quad T_{\max} = \frac{1}{2}\omega^2\int_{\alpha a}^{a}\int_{\beta b}^{b}\rho h(x,y)W^2\,\mathrm{d}y\,\mathrm{d}x \tag{3}$$

where W is the modal function of the plate, ω is the radian eigenfrequency of the structure and ρ is the material density of the plate. Defining next non-dimensional coordinates

$$\xi = x/a, \quad \eta = y/b \tag{4}$$

and substituting eqns (1), (2) and (4) into eqn (3), one has

$$U_{\max} = \frac{b}{2a^3}D_0\int_{\alpha}^{1}\int_{\beta}^{1}\xi^{3s}\eta^{3t}\left\{\left(\frac{\partial^2 W}{\partial \xi^2}\right)^2 + 2\gamma^2\frac{\partial^2 W}{\partial \xi^2}\frac{\partial^2 W}{\partial \eta^2} + \gamma^4\left(\frac{\partial^2 W}{\partial \eta^2}\right)^2 - 2(1-\nu)\gamma^2\left[\frac{\partial^2 W}{\partial \xi^2}\frac{\partial^2 W}{\partial \eta^2} - \left(\frac{\partial^2 W}{\partial \xi\,\partial \eta}\right)^2\right]\right\}\mathrm{d}\eta\,\mathrm{d}\xi, \quad T_{\max} = \frac{ab}{2}\rho h_0\omega^2\int_{\alpha}^{1}\int_{\beta}^{1}\xi^s\eta^t W^2\,\mathrm{d}\eta\,\mathrm{d}\xi \tag{5}$$

in which, $\gamma = a/b = \Gamma(1-\beta)/(1-\alpha)$ where $\Gamma = A/B$ is the aspect ratio of the plate. It is assumed that the variables in the modal function $W(\xi,\eta)$ are separable and may be expressed in terms of a series

$$W(\xi,\eta) = \sum_{m=M_0}^{\infty}\sum_{n=N_0}^{\infty} A_{mn}\varphi_m(\xi)\psi_n(\eta) \tag{6}$$

where $\varphi_m(\xi)$ and $\psi_n(\eta)$ are the appropriate admissible functions which satisfy at least the geometric boundary conditions, and if possible, all the boundary conditions in the Rayleigh–Ritz method. A_{mn} are the unknown coefficients. M_0 and N_0 are the beginning orders of the admissible functions $\varphi_m(\xi)$ and $\psi_n(\eta)$, respectively, and are decided by the practical case to be investigated.

Substituting eqn (6) into eqn (5) and minimizing the total potential energy of the plate with respect to the coefficients A_{mn} as follows

$$\frac{\partial}{\partial A_{mn}}(U_{\max}-T_{\max}) = 0 \tag{7}$$

will lead to the next eigenfrequency equation

$$\sum_{m=M_0}^{\infty}\sum_{n=N_0}^{\infty}[(1-\alpha)^4 C_{mnij}-\Omega^2\bar{E}_{mi}\bar{F}_{nj}]A_{mn} = 0, \quad i, = M_0, M_0+1, M_0+2,\ldots,\infty$$

$$j = N_0, \quad N_0+1, \quad N_0+2,\ldots,\infty \tag{8}$$

where

$$C_{mnij} = E_{mi}^{(2,2)}F_{nj}^{(0,0)}+2\gamma^2(1-\nu)E_{mi}^{(1,1)}F_{nj}^{(1,1)}+\gamma^4E_{mi}^{(0,0)}F_{nj}^{(2,2)}$$
$$+\nu\gamma^2(E_{mi}^{(0,2)}F_{nj}^{(2,0)}+E_{mi}^{(2,0)}F_{nj}^{(0,2)}), \quad \Omega^2 = \rho h_0\omega^2A^4/D_0$$

$$\bar{E}_{mi} = \int_\alpha^1 \xi^s\varphi_m\varphi_i\,\mathrm{d}\xi, \quad \bar{F}_{nj} = \int_\beta^1 \eta^t\psi_n\psi_j\,\mathrm{d}\eta \tag{9}$$

in which,

$$E_{mi}^{(p,q)} = \int_\alpha^1 \xi^{3s}(\mathrm{d}^p\varphi_m/\mathrm{d}\xi^p)(\mathrm{d}^q\varphi_i/\mathrm{d}\xi^q)\,\mathrm{d}\xi$$

$$F_{nj}^{(p,q)} = \int_\beta^1 \eta^{3t}(\mathrm{d}^p\psi_n/\mathrm{d}\eta^p)(\mathrm{d}^q\psi_j/\mathrm{d}\eta^q)\,\mathrm{d}\eta, \quad p,q = 0,1,2 \tag{10}$$

Truncating m, n, i, j in eqn (8), the solution yields the eigenfrequencies of the free vibration of the plate together with the coefficients for the modal shape (6).

3. A set of static beam functions

A unit width of strip is taken out as a beam with the same variation of the depth as the rectangular plate in one or the other direction. Without loss of generality, only a strip in the x direction is considered here because the tapered plate investigated is with similar thickness variations in both directions. It is well known that the static deflection z of the non-uniform beam under a static load $q(x)$ must satisfy the governing differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI(x)\frac{\mathrm{d}^2z}{\mathrm{d}x^2}\right) = \sum_{k=1}^{K_x} p_k\delta(x-x_k)+q(x) \tag{11}$$

where $EI = EI_0(x/a)^{3s}$ is the flexural rigidity of the beam and EI_0 is that at $x = 0$. p_k are the reaction forces of the kth intermediate point supports of the beam. $\delta(x-x_k)$ are the Dirac delta functions. Taking $\xi_k = x_k/a$, $P_k = p_k a^3/EI_0$ and $Q(\xi) = q(a\xi)a^4/EI_0$, eqn (11) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(\xi^{3s}\frac{\mathrm{d}^2 z}{\mathrm{d}\xi^2}\right) = \sum_{k=1}^{K_x} P_k\delta(\xi-\xi_k)+Q(\xi) \tag{12}$$

Correspondingly, the intermediate support conditions of the beam are

$$z(\xi_k) = 0, \quad k = 1, 2, \ldots, K_x \tag{13}$$

and the boundary conditions of the beam are

$$(L_1 z)_{\xi=\alpha} = 0, \quad (L_2 z)_{\xi=\alpha} = 0, \quad (L_3 z)_{\xi=1} = 0, \quad (L_4 z)_{\xi=1} = 0 \tag{14}$$

where L_j ($j = 1, 2, 3, 4$) are the differential operator describing the boundary conditions of the beam. For example, if the beam is clamped at the left end one has $L_1 = 1$, $L_2 = \mathrm{d}/\mathrm{d}\xi$, if the beam is simply supported at the left end one has $L_1 = 1$, $L_2 = \xi^{3s}\,\mathrm{d}^2/\mathrm{d}\xi^2$ and if the beam is free at the left end one has $L_1 = \xi^{3s}\,\mathrm{d}^2/\mathrm{d}\xi^2$, $L^2 = \mathrm{d}(\xi^{3s}\,\mathrm{d}^2/\mathrm{d}\xi^2)/\mathrm{d}\xi$. Identically, the differential operators L_3 and L_4 can also be given according to the boundary conditions of the beam at the right end.

An arbitrary load $Q(\xi)$ can be expanded into a Taylor series as follows

$$Q(\xi) = \sum_{i=0}^{\infty} Q_i(\xi-\xi_c)^i = \sum_{i=0}^{\infty} Q_i \sum_{k=0}^{i}(-1)^{i-k}C_k^i\xi_c^{i-k}\xi^k \tag{15}$$

where Q_i are the undetermined constants which may be decided uniquely if $Q(\xi)$ is given. ξ_c is the expanding point of the Taylor series and $C_k^i = i!/k!(i-k)!$.

Substituting eqn (15) into eqn (12), the static solution of the tapered beam may be written in the form of

$$z(\xi) = \sum_{i=0}^{\infty} Q_i z_i(\xi) \tag{16}$$

According to the theory of linear differential equation, the general solution of eqn (12) must be made up of two parts: homogeneous solution and special solution for every i, i.e.

$$z_i(\xi) = \bar{z}_i(\xi)+\tilde{z}_i(\xi) \tag{17}$$

and the special solution $\tilde{z}_i(\xi)$ may be further written as follows

$$\tilde{z}_i(\xi) = \sum_{k=1}^{K_x} P_k^i\tilde{z}_p^k(\xi)+\sum_{k=0}^{i}\bar{C}_k^i\tilde{z}_q^k(\xi) \tag{18}$$

where $P_k^i = P_k/Q_i$ and $\bar{C}_k^i = (-1)^{i-k}C_k^i\xi_c^{i-k}/(k+1)(k+2)$.

Solving differential equation (12), the homogeneous solution $\bar{z}_i(\xi)$ are obtained as follows

$$\bar{z}_i(\xi) = b_0^i+b_1^i\xi+b_2^i\xi^{-3s+2}+b_3^i\xi^{-3s+3}, \quad \text{for } s \neq 1/3, 2/3, 1$$

$$\bar{z}_i(\xi) = b_0^i+b_1^i\xi+b_2^i\xi(\ln\xi-1)+b_3^i\xi^2, \quad \text{for } s = 1/3$$

$$\bar{z}_i(\xi) = b_0^i + b_1^i \xi + b_2^i \ln \xi + b_3^i \xi(\ln \xi - 1), \quad \text{for } s = 2/3$$

$$\bar{z}_i(\xi) = b_0^i + b_1^i \xi + b_2^i / \xi + b_3^i \ln \xi, \quad \text{for } s = 1 \tag{19}$$

where $b_j^i (j = 0, 1, 2, 3)$ are the unknown constants. The special solution $\tilde{z}_p^k(\xi)$ are obtained as follows

$$\tilde{z}_p^k(\xi) = \frac{1}{2-3s}\{\xi^{3-3s}/(3-3s) - \xi_k \xi^{2-3s}/(1-3s) + \xi_k^{2-3s}\xi/(1-3s)$$

$$-\xi_k^{3-3s}/(3-3s)\} U(\xi - \xi_k), \quad \text{for } s \neq 1/3, 2/3, 1$$

$$\tilde{z}_p^k(\xi) = \{(\xi^2 - \xi_k^2)/2 - \xi_k \xi \ln(\xi/\xi_k)\} U(\xi - \xi_k), \quad \text{for } s = 1/3$$

$$\tilde{z}_p^k(\xi) = \{(\xi + \xi_k) \ln(\xi/\xi_k) - 2(\xi - \xi_k)\} U(\xi - \xi_k), \quad \text{for } s = 2/3$$

$$\tilde{z}_p^k(\xi) = \{\ln(\xi_k/\xi) - \xi_k/2\xi + \xi/2\xi_k\} U(\xi - \xi_k), \quad \text{for } s = 1 \tag{20}$$

where $U(\xi - \xi_k)$ are the Heaviside functions, and the special solution $\tilde{z}_q^k(\xi)$ are also obtained as follows

$$\tilde{z}_q^k(\xi) = \xi^{k+4-3s}/(k+3-3s)(k+4-3s), \quad \text{for } k \neq 3s-3, 3s-4$$

$$\tilde{z}_q^k(\xi) = \xi(\ln \xi - 1), \quad \text{for } k = 3s-3$$

$$\tilde{z}_q^k(\xi) = -\ln \xi, \quad \text{for } k = 3s-4 \tag{21}$$

3.1. *The truncated beam*

For a truncated tapered beam without rigid body motion, the reaction forces $P_k^i (k = 1, 2, \ldots, K_x)$ of the intermediate point supports and the unknown constants $b_j^i (j = 0, 1, 2, 3)$ may be uniquely decided by the boundary conditions and the zero displacement conditions at the intermediate point supports and the solution may be written in the matrix form of

$$\begin{bmatrix} A & D \\ F & G \end{bmatrix} \begin{bmatrix} B^i \\ P^i \end{bmatrix} = \begin{bmatrix} -R^i \\ -S^i \end{bmatrix} \tag{22}$$

where A is a $K_x \times 4$ matrix, D is a $K_x \times K_x$ matrix and R^i is a $K_x \times 1$ matrix, which correspond, respectively, to the values of $\bar{z}_i(\xi)$, $\tilde{z}_p^k(\xi) (k = 1, 2, \ldots, K_x)$ and $\sum_{k=0}^{i} \bar{C}_k^i \tilde{z}_q^k(\xi)$ at the intermediate point supports of the beam and can be, respectively, written as follows

$$A = \begin{bmatrix} 1 & \xi_1 & \xi_1^{-3s+2} & \xi_1^{-3s+3} \\ 1 & \xi_2 & \xi_2^{-3s+2} & \xi_2^{-3s+3} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \xi_{K_x} & \xi_{K_x}^{-3s+2} & \xi_{K_x}^{-3s+3} \end{bmatrix}, \quad \text{for } s \neq 1/3, 2/3, 1$$

$$A = \begin{bmatrix} 1 & \xi_1 & \xi_1(\ln\xi_1 - 1) & \xi_1^2 \\ 1 & \xi_2 & \xi_2(\ln\xi_2 - 1) & \xi_2^2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \xi_{K_x} & \xi_{K_x}(\ln\xi_{K_x} - 1) & \xi_{K_x}^2 \end{bmatrix}, \quad \text{for } s = 1/3$$

$$A = \begin{bmatrix} 1 & \xi_1 & \ln\xi_1 & \xi_1(\ln\xi_1 - 1) \\ 1 & \xi_2 & \ln\xi_2 & \xi_2(\ln\xi_2 - 1) \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \xi_{K_x} & \ln\xi_{K_x} & \xi_{K_x}(\ln\xi_{K_x} - 1) \end{bmatrix}, \quad \text{for } s = 2/3$$

$$A = \begin{bmatrix} 1 & \xi_1 & 1/\xi_1 & \ln\xi_1 \\ 1 & \xi_2 & 1/\xi_2 & \ln\xi_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \xi_{K_x} & 1/\xi_{K_x} & \ln\xi_{K_x} \end{bmatrix}, \quad \text{for } s = 1 \tag{23}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ \tilde{z}_p^1(\xi_2) & 0 & 0 & \dots & 0 \\ \tilde{z}_p^1(\xi_3) & \tilde{z}_p^2(\xi_3) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{z}_p^1(\xi_{K_x}) & \tilde{z}_p^2(\xi_{K_x}) & \dots & \tilde{z}_p^{K_x-1}(\xi_{K_x}) & 0 \end{bmatrix} \tag{24}$$

$$R^i = \begin{bmatrix} \sum_{k=0}^{i} \bar{C}_k^i \tilde{z}_q^k(\xi_1) \\ \sum_{k=0}^{i} \bar{C}_k^i \tilde{z}_q^k(\xi_2) \\ \vdots \\ \sum_{k=0}^{i} \bar{C}_k^i \tilde{z}_q^k(\xi_{K_x}) \end{bmatrix} \tag{25}$$

F is a 4×4 matrix, G is a $4 \times K_x$ matrix and S^i is a 4×1 matrix, which correspond, respectively, to those in the boundary conditions of the truncated beam. For example, for a simply–simply supported beam one has

$$F = \begin{bmatrix} 1 & \alpha & \alpha^{-3s+2} & \alpha^{-3s+3} \\ 0 & 0 & (-3s+2)(-3s+1)\alpha^{-3s} & (-3s+3)(-3s+2)\alpha^{-3s+1} \\ 1 & 1 & 1 & 1 \\ 0 & 0 & (-3s+2)(-3s+1) & (-3s+3)(-3s+2) \end{bmatrix}, \quad \text{for } s \neq 1/3, 2/3, 1$$

$$F = \begin{bmatrix} 1 & \alpha & \alpha(\ln\alpha - 1) & \alpha^2 \\ 0 & 0 & 1/\alpha & 2 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad \text{for } s = 1/3$$

$$F = \begin{bmatrix} 1 & \alpha & \ln\alpha & \alpha(\ln\alpha - 1) \\ 0 & 0 & -1/\alpha^2 & 1/\alpha \\ 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad \text{for } s = 2/3$$

$$F = \begin{bmatrix} 1 & \alpha & 1/\alpha & \ln\alpha \\ 0 & 0 & 2/\alpha^3 & -1/\alpha^2 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}, \quad \text{for } s = 1 \tag{26}$$

$$G = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \tilde{z}_p^1(1) & \tilde{z}_p^2(1) & \dots & \tilde{z}_p^{K_x}(1) \\ \ddot{\tilde{z}}_p^1(1) & \ddot{\tilde{z}}_p^2(1) & \dots & \ddot{\tilde{z}}_p^{K_x}(1) \end{bmatrix} \tag{27}$$

$$S^i = \begin{bmatrix} \sum_{k=0}^{i} \bar{C}_k^i \tilde{z}_q^k(\alpha) \\ \sum_{k=0}^{i} \bar{C}_k^i \ddot{\tilde{z}}_q^k(\alpha) \\ \sum_{k=0}^{i} \bar{C}_k^i \tilde{z}_q^k(1) \\ \sum_{k=0}^{i} \bar{C}_k^i \ddot{\tilde{z}}_q^k(1) \end{bmatrix} \tag{28}$$

where

$$\ddot{\tilde{z}}_p^k(1) = \{\mathrm{d}^2 \tilde{z}_p^k(\xi)/\mathrm{d}\xi^2\}_{\xi=1} \quad (k = 1, 2, \dots, K_x)$$

and

$$\ddot{\tilde{z}}_q^k(\alpha) = \{\mathrm{d}^2 \tilde{z}_q^k(\xi)/\mathrm{d}\xi^2\}_{\xi=\alpha}, \quad \ddot{\tilde{z}}_q^k(1) = \{\mathrm{d}^2 \tilde{z}_q^k(\xi)/\mathrm{d}\xi^2\}_{\xi=1} \quad (k = 0, 1, 2, \dots, i)$$

Similarly, the matrices F, G and S^i for other types of boundary conditions may also be given if one wishes. B^i and P^i are the unknown coefficient matrices

$$B^i = [b_0^i, b_1^i, b_2^i, b_3^i]^T, \quad P^i = [P_1^i, P_2^i, \ldots, P_{K_x}^i]^T \tag{29}$$

3.2. *The sharp ended beam*

For a sharp ended beam, the sharp end cannot sustain a bending moment or a shearing force, hence one has

$$b_2^i = 0, \quad b_3^i = 0 \tag{30}$$

and the deflection and the rotational angle of the beam should be finite at the sharp end. So there is a limit to the beginning order of the Taylor series of loads in eqn (15) as follows

$$i, k > 3s-3 \tag{31}$$

therefore, eqns (15) and (16) should be, respectively, rewritten as follows

$$Q(\xi) = \sum_{i=J_0}^{\infty} Q_i \sum_{k=J_0}^{i} (-1)^{i-k} C_k^i \xi_c^{i-k} \xi^k \tag{32}$$

$$z(\xi) = \sum_{i=J_0}^{\infty} Q_i z_i(\xi) \tag{33}$$

in which,

$$J_0 = \max\{\mathrm{Int}(3s-2), 0\} \tag{34}$$

where Int is the integer function. In this case, the matrix A is a $K_x \times 2$ matrix as follows

$$A = \begin{bmatrix} 1 & \xi_1 \\ 1 & \xi_2 \\ \vdots & \vdots \\ 1 & \xi_{K_x} \end{bmatrix} \tag{35}$$

and the matrix R^i becomes

$$R^i = \begin{bmatrix} \sum_{k=J_0}^{i} \bar{C}_k^i \bar{z}_q^k(\xi_1) \\ \sum_{k=J_0}^{i} \bar{C}_k^i \bar{z}_q^k(\xi_2) \\ \vdots \\ \sum_{k=J_0}^{i} \bar{C}_k^i \bar{z}_q^k(\xi_{K_x}) \end{bmatrix} \tag{36}$$

and F is a 2×2 matrix, G is a $2 \times K_x$ matrix and S^i is a 2×1 matrix, which are decided by the boundary conditions of the beam at the end $\xi = 1$. For example, for a cantilevered beam with a sharp end one has

$$F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{37}$$

$$G = \begin{bmatrix} \tilde{z}_p^1(1) & \tilde{z}_p^2(1) & \dots & \tilde{z}_p^{K_x}(1) \\ \dot{\tilde{z}}_p^1(1) & \dot{\tilde{z}}_p^2(1) & \dots & \dot{\tilde{z}}_p^{K_x}(1) \end{bmatrix} \tag{38}$$

$$S^i = \begin{bmatrix} \sum_{k=J_0}^{i} \bar{C}_k^i \tilde{z}_q^k(1) \\ \sum_{k=J_0}^{i} \bar{C}_k^i \dot{\tilde{z}}_q^k(1) \end{bmatrix} \tag{39}$$

Similarly, the matrices F, G and S^i for other types of boundary conditions can also be established.

3.3. *The tapered beam with rigid body motions*

By solving eqn (22), the unknown constants $b_j^i (j = 0, 1, 2, 3)$ and the reaction forces $P_k^i (k = 1, 2, \dots, K_x)$ of the intermediate point supports may be uniquely decided. However, for a beam with rigid motions, the unknown coefficients in eqn (22) cannot be decided upon by the approach described above. In this case, the total displacement of the beam may be considered as the sum of rigid body motions and the deflection of the beam. For example, for a free–free truncated beam with one intermediate point support at $\xi = \xi_1$, the rigid body rotation of the beam around the point support exists. One may rewrite eqn (16) as follows

$$z(\xi) = \sum_{i=-1}^{\infty} Q_i z_i(\xi) \tag{40}$$

in which,

$$z_{-1}(\xi) = \xi_1 - \xi \tag{41}$$

and the other $z_i(\xi)$ $(i = 0, 1, 2, \dots)$ are those of the free-simply supported (or simply supported-free) truncated beam with a corresponding intermediate point support. Similarly, for a free–free sharp ended beam with one intermediate point support at $\xi = \xi_1$, one can rewrite eqn (33) as follows

$$z(\xi) = \sum_{i=J_0-1}^{\infty} Q_i z_i(\xi) \tag{42}$$

in which,

$$z_{J_0-1}(\xi) = \xi_1 - \xi \tag{43}$$

and other $z_i(\xi)$ $(i = J_0, J_0+1, J_0+2, \dots)$ are those of the free-simply supported sharp ended beam with a corresponding intermediate point support. For the tapered beam with rigid body motions but without an intermediate point support, the handling method has been described by Zhou and Cheung (1997). It may be seen from eqn (22) that only one inverse calculation to the coefficient matrix is needed since the matrices A, D, F and G are all independent of the series variation i. This greatly reduces the computational cost.

4. Numerical examples

In order to demonstrate the accuracy, convergency and applicability of the present models, some numerical results are tabulated and compared with the values available in the literature. Four capital letters are used to represent the boundary conditions of the plate. The first two letters express the boundary conditions of the plate in the x direction and the last two express those in the y direction. S implies a simply supported edge, C a clamped edge, and F a free edge. In all the numerical computations, the expanding point of the Taylor series is taken as the midpoint of the beam, i.e. $\xi_c = (1+\alpha)/2$ and the Poisson's ratio $v = 0.3$.

The convergency study and comparison tests for a simply supported square plate with a linearly varying thickness in the x direction are carried out. The truncation factor and the taper factor of the plate are taken as $\alpha = 5/7$, $s = 1$; $\alpha = 5/6$, $s = 1$ and $\alpha = 0$, $s = 0$ (uniform plate), respectively. The number of terms of the static beam functions varies from 1–6 in each direction. The first eight dimensionless eigenfrequencies are listed in Table 1 and compared with those obtained by the differential quadrature method (Bert and Malik, 1996) and the Rayleigh–Ritz method with the one-dimensional orthogonal polynomials as the admissible functions (Bhat et al., 1990). Good agreement is achieved.

The secondary convergency study is for a two-direction, linearly tapered square plate with a mid-line support in each direction. The truncation factors are the same in both directions and are equal to 2/3. Three types of boundary conditions are considered: simply supported edges, fully clamped edges, and two opposite edges simply supported with the other two opposite edges clamped. The first eight dimensionless eigenfrequencies are listed in Table 3 for the different numbers of terms of the static beam functions.

It can be observed from Tables 2 and 3 that the convergency is very rapid and rather accurate results can be obtained, even though only one term of the static beam function is used in each direction to determine the fundamental eigenfrequency for the first example or two terms of the static beam functions in each direction are used to determine the first four eigenfrequencies for both examples.

It should be noted that the usable number of terms of the static beam functions is limited since the calculation is carried out numerically. A lot of numerical examples show that the maximum term number (in this range, the stable solutions are constantly obtained) of the static beam functions is somewhat sensitive to the expanding point ξ_c of the Taylor series. The farther the expanding point is from the midpoint of the beam, the smaller will be the maximum term number, especially for the beam with a larger truncation factor. However, if the midpoint of the beam is taken as the expanding point of the Taylor series, namely $\xi_c = (1+\alpha)/2$, the maximum term number of the static beam functions is the largest and independent of the truncation factor of the beam. In this case, the convergency rate is also the fastest.

Next, sets of first-time presented data are tabulated for the one-direction or two-direction, linearly or parabolically tapered rectangular plates with one or two intermediate line supports in one or two directions. The first four dimensionless eigenfrequencies are given. Four terms of the

Table 2
The convergency and comparison studies of the dimensionless eigenfrequencies Ω_i ($i = 1, 2, \ldots, 8$) for a simply supported square plate with a linearly varying thickness in one direction

α, s	$m \times n$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7	Ω_8
5/7, 1	1×1	16.909							
	2×2	16.875	42.221	42.350	67.999				
	3×3	16.867	42.221	42.339	67.800	85.353	86.145	111.55	111.64
	4×4	16.867	41.984	42.097	67.430	85.353	86.127	111.19	111.54
	5×5	16.867	41.984	42.096	67.425	83.414	84.142	109.46	109.73
	6×6	16.867	41.983	42.096	67.424	83.414	84.135	109.44	109.73
	Bert (1996)	16.864	41.978	42.090	67.411	83.382	84.104		
	Bhat (1990)	16.864	40.549	42.092	67.412	83.886	84.368		
5/6, 1	1×1	18.099							
	2×2	18.088	45.398	45.439	72.776				
	3×3	18.080	45.396	45.434	72.712	92.235	94.491	119.60	119.64
	4×4	18.080	45.140	45.176	72.309	92.235	92.485	119.37	119.49
	5×5	18.080	45.140	45.176	72.307	90.116	90.351	117.48	117.57
	6×6	18.080	45.139	45.175	72.306	90.116	90.349	117.47	117.57
	Bert (1996)	18.077	45.134	45.170	72.293	90.082	90.315		
	Bhat (1990)	18.077	45.134	45.171	72.294	90.634	90.777		
0, 0	1×1	19.751							
	2×2	19.751	49.638	49.638	79.416				
	3×3	19.743	49.636	49.636	79.416				
	4×4	19.743	49.355	49.355	78.972	101.07	101.07	130.43	130.43
	5×5	19.743	49.355	49.355	79.972	98.733	98.733	128.35	128.35
	6×6	19.743	49.354	49.354	78.971	98.733	98.733	128.35	128.35
	Exact	19.739	49.348	49.348	78.957	98.696	98.696	128.30	128.30

Table 3
The convergence study of the dimensionless eigenfrequencies Ω_i ($i = 1, 2, \ldots, 8$) for a two-direction, linearly tapered square plate with a mid-line support in each direction and the same truncation factors in both directions

Edges	$m \times n$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7	Ω_8
SS–SS	1×1	66.380							
	2×2	50.335	64.027	65.149	81.992				
	3×3	50.320	63.992	65.108	81.907	140.94	141.29	174.36	174.56
	4×4	50.223	63.711	64.861	81.290	123.72	123.79	152.14	152.33
	5×5	50.233	63.705	64.855	81.276	123.27	123.29	151.53	151.78
	6×6	50.222	63.694	64.847	81.250	119.99	120.03	147.55	147.22
	7×7	50.222	63.693	64.846	81.247	119.95	119.99	147.49	147.66
CC–CC	1×1	86.484							
	2×2	68.119	86.309	87.373	109.21				
	3×3	68.113	86.283	87.359	109.17	172.76	173.04	209.73	209.73
	4×4	68.058	86.030	87.160	108.62	154.23	154.80	187.48	187.92
	5×5	68.054	86.012	87.142	108.59	152.88	153.49	185.78	186.41
	6×6	68.049	85.968	87.101	108.48	147.47	148.01	179.15	179.78
	7×7	68.049	85.967	87.099	108.47	147.43	147.98	179.11	179.74
SC–CC	1×1	77.091							
	2×2	59.598	74.967	77.806	96.974				
	3×3	59.583	74.928	77.784	96.914	147.84	166.58	179.68	205.88
	4×4	59.506	74.580	77.599	96.278	130.29	148.54	158.78	164.08
	5×5	59.505	74.573	77.584	96.255	129.79	147.21	158.14	163.87
	6×6	59.501	74.555	77.554	96.194	126.77	141.44	154.43	161.27
	7×7	59.501	74.554	77.551	96.190	126.72	141.41	154.38	160.43

Table 4
The dimensionless eigenfrequencies Ω_i ($i = 1, 2, 3, 4$) for a one-direction, linearly tapered rectangular plate with a mid-line support in the taper direction

α	Edges	Ω_1	Ω_2	Ω_3	Ω_4
0.5	SS–SS	53.445	75.788	124.37	136.91
	CC–CC	84.501	117.55	174.82	179.68
	FC–FF	8.5094	21.681	44.899	66.695
	FS–FF	7.9323	21.556	36.044	55.199
0.6	SS–SS	59.308	79.001	140.13	152.68
	CC–CC	93.900	122.11	195.92	202.14
	FC–FF	8.7546	23.475	48.426	73.012
	FS–FF	8.1318	23.326	38.161	59.708
0.7	SS–SS	64.871	82.386	155.59	167.86
	CC–CC	103.36	126.78	216.54	224.15
	FC–FF	9.0142	25.257	51.754	77.053
	FS–FF	8.3419	25.084	40.186	64.204
0.9	SS–SS	70.053	86.050	170.73	182.28
	CC–CC	112.29	131.82	235.66	245.75
	FC–FF	9.2810	27.027	54.899	80.790
	FS–FF	8.5602	26.831	42.141	67.929
0.9	SS–SS	74.770	90.110	185.22	195.22
	CC–CC	120.40	137.43	254.52	266.61
	FF–FF	9.5518	28.787	57.874	84.556
	FS–FF	8.7837	28.568	44.041	70.251

Table 5
The dimensionless eigenfrequencies Ω_i ($i = 1, 2, 3, 4$) for a one-direction, linearly tapered rectangular plate with a line support at one third of the side length from the small thickness edge in the taper direction

α	Edges	Ω_1	Ω_2	Ω_3	Ω_4
0.5	SS–SS	53.176	86.739	118.77	146.27
	CC–CC	90.245	124.83	157.63	206.48
	FC–FF	13.827	30.725	36.115	50.551
	FS–FF	11.655	24.925	30.025	42.249
0.6	SS–SS	55.710	97.127	125.09	155.89
	CC–CC	94.802	140.04	166.39	226.50
	FC–FF	14.485	33.897	38.012	53.173
	FS–FF	12.144	26.670	32.900	44.742
0.7	SS–SS	58.156	106.78	132.07	163.09
	CC–CC	99.148	153.06	177.32	238.54
	FC–FF	15.136	37.024	39.873	55.803
	FS–FF	12.634	28.389	35.650	47.354
0.8	SS–SS	60.538	115.49	139.95	169.70
	CC–CC	103.35	163.41	190.87	249.08
	FC–FF	15.776	40.098	41.701	58.457
	FS–FF	13.122	30.084	38.264	50.100
0.9	SS–SS	62.869	123.15	148.79	176.00
	CC–CC	107.43	171.82	206.26	259.01
	FC–FF	16.406	43.111	43.504	61.147
	FS–FF	13.604	31.757	40.740	52.981

Table 6
The dimensionless eigenfrequencies Ω_i ($i = 1, 2, 3, 4$) for a one-direction, parabolically tapered rectangular plate with a mid-line support in the taper direction

α	Edges	Ω_1	Ω_2	Ω_3	Ω_4
α	SS–SS	34.403	63.794	75.846	87.435
	CC–CC	52.814	97.474	109.53	112.40
	FC–FF	7.1049	15.247	31.688	41.128
	FS–FF	6.7313	15.187	27.421	39.478
0.6	SS–SS	42.698	68.891	96.862	108.76
	CC–CC	66.389	106.94	138.40	139.97
	FC–FF	7.6448	17.955	37.498	52.295
	FS–FF	7.1866	17.868	31.249	46.317
0.7	SS–SS	51.647	74.078	119.96	132.27
	CC–CC	81.147	114.92	169.02	173.17
	FC–FF	8.1875	20.839	43.404	64.488
	FS–FF	7.6426	20.719	34.957	53.310
0.8	SS–SS	61.042	79.708	145.06	157.48
	CC–CC	96.874	123.01	202.58	209.04
	FC–FF	8.7322	23.898	49.303	74.159
	FS–FF	8.0987	23.741	38.606	60.863
0.9	SS–SS	70.416	86.284	171.89	183.22
	CC–CC	112.92	132.07	237.82	247.35
	FC–FF	9.2781	27.131	55.091	81.015
	FS–FF	8.5547	26.933	42.242	68.060

Table 7
The dimensionless eigenfrequencies Ω_i ($i = 1, 2, 3, 4$) for a sharp ended, linearly tapered rectangular plate with one intermediate line support in the taper direction

ξ_1	Edges	Ω_1	Ω_2	Ω_3	Ω_4
1/4	FC–FF	15.093	20.572	21.982	30.019
	FS–FF	8.9525	18.620	20.141	26.174
1/2	FC–FF	9.2274	14.341	25.678	27.799
	FS–FF	8.8229	14.304	24.220	26.503
3/4	FC–FF	6.5636	12.536	18.886	25.417
	FS–FF	6.4360	12.508	18.578	25.314

Table 8
The dimensionless eigenfrequencies Ω_i ($i = 1, 2, 3, 4$) for a two-direction, linearly tapered square plate with a mid-line support in each direction and the same truncation factors in both directions

$\alpha = \beta$	Edges	Ω_1	Ω_2	Ω_3	Ω_4
0.5	SS–SS	35.878	50.013	51.294	71.407
	CC–CC	48.140	67.425	68.651	95.563
	FC–FC	9.9755	30.082	33.315	43.427
	FS–FS	9.7514	26.613	27.641	35.469
	FF–FF	7.6954	11.089	12.384	16.653
0.6	SS–SS	44.350	58.081	59.333	77.204
	CC–CC	59.873	78.425	79.634	103.25
	FC–FC	11.945	32.243	40.888	52.258
	FS–FS	11.629	31.866	32.912	43.873
	FF–FF	8.7140	13.053	13.940	18.515
0.7	SS–SS	53.197	66.610	67.678	83.426
	CC–CC	72.220	89.926	90.987	111.42
	FC–FC	14.070	44.904	48.796	61.849
	FS–FS	13.651	37.381	38.479	53.226
	FF–FF	9.8043	15.119	15.637	20.542
0.8	SS–SS	62.119	75.650	76.337	90.360
	CC–CC	84.762	102.01	102.73	120.48
	FC–FC	16.350	52.998	57.008	72.241
	FS–FS	15.817	43.174	44.345	63.490
	FF–FF	10.971	17.245	17.470	22.767
0.9	SS–SS	70.790	85.093	85.316	98.454
	CC–CC	96.948	114.62	114.87	131.08
	FC–FC	18.782	61.472	65.512	83.469
	FS–FS	18.124	49.252	50.510	74.637
	FF–FF	12.214	19.396	19.438	25.223

Table 9
The dimensionless eigenfrequencies Ω_i ($i = 1, 2, 3, 4$) for a two-direction, linearly tapered square plate with a line support at one third of the side length from the small thickness edge in each direction and the same truncation factors in both directions

$\alpha = \beta$	Edges	Ω_1	Ω_2	Ω_3	Ω_4
0.5	SS–SS	34.422	63.487	64.354	73.347
	CC–CC	46.931	85.433	86.192	95.523
	FC–FC	17.444	28.615	28.774	48.452
	FS–FS	16.930	24.025	24.523	37.095
	FF–FF	5.0734	14.944	15.149	19.222
0.6	SS–SS	37.904	74.750	75.013	93.309
	CC–CC	51.672	100.33	100.43	121.25
	FC–FC	21.594	34.207	34.274	53.423
	FS–FS	20.638	28.694	28.787	41.054
	FF–FF	5.8231	16.750	17.190	23.790
0.7	SS–SS	41.445	85.438	85.535	108.11
	CC–CC	56.470	113.22	113.59	137.79
	FC–FC	26.082	40.004	40.375	58.566
	FS–FS	24.512	33.362	33.840	45.259
	FF–FF	6.6542	18.720	19.269	28.906
0.8	SS–SS	45.057	95.716	95.755	119.61
	CC–CC	61.343	125.02	125.45	155.92
	FC–FC	30.889	46.153	46.888	63.926
	FS–FS	28.535	38.239	39.384	49.802
	FF–FF	7.5631	20.851	21.442	34.496
0.9	SS–SS	48.745	105.61	105.62	132.04
	CC–CC	66.300	136.24	136.62	175.65
	FC–FC	35.995	52.664	53.767	69.561
	FS–FS	32.698	43.414	45.184	54.830
	FF–FF	8.5458	23.139	23.729	40.526

Table 10
The dimensionless eigenfrequencies Ω_i ($i = 1, 2, 3, 4$) for a two-direction, linearly tapered square plate with a mid-line support in each direction and the unequal truncation factors in two directions

α, β	Edges	Ω_1	Ω_2	Ω_3	Ω_4
0.5, 0.6	SS–SS	39.901	52.603	56.500	74.263
	CC–CC	53.690	70.706	76.034	99.342
	FC–FC	10.928	33.343	37.043	46.238
	FS–FS	10.661	28.545	30.744	39.285
	FF–FF	8.1921	12.034	13.125	17.570
0.6, 0.7	SS–SS	48.586	61.034	64.547	80.272
	CC–CC	65.764	82.144	87.007	107.27
	FC–FC	12.974	40.726	44.824	55.797
	FS–FS	12.609	33.874	36.237	48.211
	FF–FF	9.2443	14.054	14.752	19.510
0.7, 0.8	SS–SS	57.498	69.998	72.858	86.845
	CC–CC	78.249	94.208	98.267	115.88
	FC–FC	15.175	48.565	52.942	66.090
	FS–FS	14.702	39.502	41.993	58.046
	FF–FF	10.372	16.154	16.518	21.630
0.8, 0.9	SS–SS	66.321	79.524	81.398	94.336
	CC–CC	90.657	107.00	109.76	125.69
	FC–FC	17.531	56.801	61.372	77.126
	FS–FS	16.937	45.429	48.026	68.769
	FF–FF	11.576	18.296	18.419	23.964

Table 11
The dimensionless eigenfrequencies Ω_i ($i = 1, 2, 3, 4$) for a one-direction, linearly tapered rectangular plate with two equally spaced intermediate line supports in the taper direction

α	Edges	Ω_1	Ω_2	Ω_3	Ω_4
0.5	SS–SS	112.05	153.91	187.03	262.95
	CC–CC	176.66	216.67	280.27	373.65
	FC–FF	15.989	43.890	82.020	116.53
	FS–FF	15.903	43.885	74.921	100.56
	FF–FF	13.414	18.226	43.819	58.593
0.6	SS–SS	126.96	162.88	194.39	301.31
	CC–CC	199.22	232.54	286.93	429.04
	FC–FF	17.038	48.829	88.969	127.25
	FS–FF	16.939	48.821	79.614	108.63
	FF–FF	14.277	19.461	48.719	60.645
0.7	SS–SS	141.37	170.66	202.81	339.24
	CC–CC	221.28	249.20	293.93	484.37
	FC–FF	18.096	53.748	95.484	137.74
	FS–FF	17.985	53.738	83.852	116.32
	FF–FF	15.149	20.694	53.575	62.710
0.8	SS–SS	155.05	178.10	213.32	376.62
	CC–CC	241.06	267.19	301.62	539.54
	FC–FF	19.158	58.652	101.65	147.76
	FS–FF	19.035	58.640	87.761	123.69
	FF–FF	16.028	21.921	58.358	64.821

Table 12
The dimensionless eigenfrequencies Ω_i ($i = 1, 2, 3, 4$) for a two-direction, linearly tapered square plate with two equally spaced intermediate line supports in each direction and the same truncation factors in both directions

$\alpha = \beta$	Edges	Ω_1	Ω_2	Ω_3	Ω_4
0.5	SS–SS	70.120	95.330	96.666	115.39
	CC–CC	92.375	112.53	115.63	140.55
	FC–FC	18.461	54.468	57.481	71.134
	FS–FS	18.396	26.269	52.901	57.118
	FF–FF	18.348	25.944	26.262	35.996
0.6	SS–SS	90.283	114.72	116.03	136.08
	CC–CC	117.34	137.22	140.51	163.05
	FC–FC	23.198	68.539	71.552	90.400
	FS–FS	23.166	64.463	65.847	80.713
	FF–FF	22.974	29.689	30.044	38.383
0.7	SS–SS	112.19	134.52	135.71	158.45
	CC–CC	142.28	164.26	166.88	188.63
	FC–FC	28.444	83.589	86.606	111.79
	FS–FS	28.399	76.951	78.145	98.146
	FF–FF	28.012	33.671	34.041	40.885
0.8	SS–SS	135.17	154.47	155.40	176.83
	CC–CC	166.07	192.96	194.30	217.91
	FC–FC	34.197	99.577	102.61	134.90
	FS–FS	34.136	89.837	90.916	116.79
	FF–FF	33.342	37.921	38.250	43.585

static beam functions are used in the direction with one or no intermediate line support and five terms of the static beam functions in the direction with two intermediate line supports.

First a one-direction, tapered rectangular plate with one intermediate line support in the taper direction is investigated. The aspect ratio of the plate is $\Gamma = 2$. The results for the linearly tapered plate with a line support, respectively, at the middle and one third of the side length from the small thickness edge are listed in Tables 4 and 5, and the results for the parabolically tapered plate with a mid-line support are listed in Table 6. The combinations of four types of boundary conditions: SS–SS, CC–CC, FC–FF and FS–FF and five different truncation factors varying from 0.5–0.9 are considered. Also the results for a sharp ended plate with a linear taper and one intermediate line support are listed in Table 7. The combinations of two types of boundary conditions: FC–FF and FS–FF and three different locations of the line support: the middle, one third and two thirds of the side length from the sharp edge are considered.

Secondly a two-direction, linearly tapered square plate with one intermediate line support in each direction is investigated. The locations of the line supports are symmetric in both directions. The results for the plate with a line support, respectively, at the middle and one third of the side length from the small thickness edge in each direction are listed in Tables 8 and 9. The combinations of five types of boundary conditions: SS–SS, CC–CC, FC–FC, FS–FS and FF–FF and five different truncation factors varying from 0.5–0.9 in both directions are considered. The results for the plate with a mid-line support in each direction but with different truncation factors in two directions are listed in Table 10. The combinations of five types of boundary conditions and four groups of different truncation factors: 0.5, 0.6; 0.6, 0.7; 0.7, 0.8 and 0.8, 0.9 are considered.

Thirdly, a one-direction, linearly tapered rectangular plate with two equally spaced intermediate line supports in the taper direction is investigated. The aspect ratio of the plate is $\Gamma = 3$. The results for the combinations of five types of boundary conditions: SS–SS, CC–CC, FC–FF, FS–FF and FF–FF and four different truncation factors varying from 0.5–0.8 are listed in Table 11.

Finally, a two-direction, linearly tapered square plate with two equally spaced intermediate line supports in each direction is investigated. The truncation factors are the same in both directions. The results for the combinations of five types of boundary conditions: SS–SS, CC–CC, FC–FC, FS–FS and FF–FF and four different truncation factors varying from 0.5–0.8 are listed in Table 12.

It can be seen from the tables that the eigenfrequencies always increase with the increase in the truncation factors of the plate. This behavior is expected because with the increase in the truncation factors, the rigidity of the plate also increases.

5. Conclusions

A new set of admissible functions is developed from the static solution of a tapered beam with intermediate point supports under a Taylor series of loads and applied to analyze the free vibration of tapered rectangular plates with intermediate line supports in one or two directions. Consistent convergency can be obtained by taking the midpoint of the beam as the expanding point of the Taylor series. The Rayleigh–Ritz method is adopted to derive the eigenfrequency equation. The convergency studies show that good accuracy can be obtained and only a small number of terms of the static beam functions needs to be used. Sets of the first-time presented results are tabulated for the one-direction or two-direction tapered rectangular plates with one or two intermediate line supports in one or two directions in Tables 3–12. The numerical results provide valuable information for engineers in design applications and may also serve as benchmarks for further reference.

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